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BORN APPROXIMATION OF THE SOLUTION OF THE INTERNAL WAVE SCATTERING PROBLEM

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Difficulties in the execution of detailed and extensive measurements of internal wave parameters in the ocean retard, to a certain extent, the development of a correctly deduced theory. In particular, little is known about the internal wave energy distribution between different modes. For a number of reasons it is considered that, for a sufficiently definite thermocline, the lowest mode will dominate, whose behavior is investigated in most detail in theoretical respects [1]. However, higher modes characterized by higher values of the transverse velocity gradient, and increase in the possibility of local instability and degeneration into turbulence, play an important part in the development of internal wave spectra. Consequently, it is of interest to examine methods of energy transmission in the internal wave spectrum. The modal structure is evidently shaped as a function of the variability of a whole series of parameters specifying the propagation law and the interaction of internal waves in the ocean. Consequently, for instance, problems of internal wave propagation in the presence of horizontal density field inhomogeneities [2, 3], shear flows [4, 5], and arbitrary vertical density field [6], etc., were examined. A sufficiently complete list of literature can be found in [7-9].

One of the possible mechanisms of internal wave energy redistribution between different modes of the scattering of internal waves by localized density field inhomogeneities is discussed. The simplest problem is formulated here: The Boussinesq approximation is used to describe a stratified fluid, and rotation of the earth is neglected while the density field inhomogeneities are considered not to vary in time and to be at rest.

Within the framework of assumptions made in the linear formulation of the problem, and neglecting molecular viscosity forces, the initial system of equations describing the dynamic state of the medium has the form [1]

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0} g\mathbf{k} = 0, \quad \nabla \mathbf{U} = 0, \quad \frac{\partial \rho}{\partial t} + \mathbf{U}\nabla\rho_1 - \rho_0 g^{-1} N_0^2 w = 0, \tag{1}$$

where $\mathbf{U} \equiv \{u, v, w\}$ is the velocity vector of particles of the medium; p, pressure; ρ, deviation of the density from the initial density distribution, equal to $\bar{\rho}(z) + \rho_1(\mathbf{r})$, where $\bar{\rho}(z)$ is the density distribution in the absence of inhomogeneities and $\rho_1(\mathbf{r})$ is a function characterizing the density-field inhomogeneity; $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, unit vectors along the Cartesian x, y, z coordinate axes; g, acceleration of gravity; $N^2_0 \equiv -(g/\rho_0)(d\bar{\rho}/dz)$, Vaisala–Brunt frequency. It should be noted that stationary flows generally exist for such an assignment of the density field. But since these flows are sufficiently slow, they can be neglected in a first approximation and a density field given by a function independent of the time can be considered (see [3], for example).

We will be interested below in a function $\mathbf{U}(\mathbf{r}, t)$. Consequently, we go from system (1) over to a system of equations for u, v, w that does not contain the functions p(x, y, z) and ρ(x, y, z):

$$\frac{\partial^2}{\partial t^2}\Delta w + N_0^2(z)\Delta_h w = \frac{g}{\rho_0}\Delta_h(\mathbf{U}\nabla\rho_1). \tag{2}$$

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$$\frac{\partial^2}{\partial t^2}\Delta u = \frac{\partial^2 [N_0^2 w]}{\partial x \partial z} - \frac{g}{\rho_0}\frac{\partial^2 [\mathbf{U}\nabla\rho_1]}{\partial x \partial z},$$
$$\frac{\partial^2}{\partial t^2}\Delta v = \frac{\partial^2 [N_0^2 w]}{\partial y \partial z} - \frac{g}{\rho_0}\frac{\partial^2 [\mathbf{U}\nabla\rho_1]}{\partial y \partial z}, \quad (2)$$

where $\Delta_h = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator in the horizontal coordinates. No general methods exist for solving system (2) for arbitrary functions $N^2{}_0$ and $\rho_1(\mathbf{r})$. In this connection, approximate methods, of which the perturbation method is most widespread, must be utilized.

We shall consider that

$$|\beta(\mathbf{r})| \equiv \left| -\frac{g}{\rho_0}\frac{\nabla\rho_1}{N_0^2} \right| \ll 1,$$

and shall seek the solution of system (2) in the form of a perturbation theory series in the small parameter $|\boldsymbol{\beta}(\mathbf{r})|$. To do this, it is convenient to go from the system of differential equations (2) to its equivalent system of integral equations. To this end, we represent the solution of system (2) in the form

$$\mathbf{U} = \int_{-\infty}^{\infty} \mathbf{Q}(\mathbf{r}, \nu)\exp(i\nu t)\,d\nu, \quad (3)$$

where Q = {U, V, W} and the spectral components of U, V, W satisfy the following system of equations:

$$\Delta W - \frac{N_0^2}{\nu^2}\Delta_h W = -\frac{g}{\rho_0 \nu^2}\Delta_h(\mathbf{Q}\nabla\rho_1), \quad (4)$$
$$\Delta U = -\frac{1}{\nu^2}\frac{\partial^2(N_0^2 W)}{\partial x \partial z} + g\rho_0^{-1}\nu^{-2}\frac{\partial^2(\mathbf{Q}\nabla\rho_1)}{\partial x \partial z},$$
$$\Delta V = -\frac{1}{\nu^2}\frac{\partial^2(N_0^2 W)}{\partial y \partial z} + g\rho_0^{-1}\nu^{-2}\frac{\partial^2(\mathbf{Q}\nabla\rho_1)}{\partial y \partial z}.$$

Let $\mathbf{Q}_0(\mathbf{r})$ be a primary wave field satisfying system (4) with $\nabla\rho_1 \equiv 0$, while $G(\mathbf{r}, \mathbf{r}')$ is the Green's function of the first equation of this system, i.e.,

$$\Delta G(\mathbf{r}, \mathbf{r}') - \beta_0^2 \Delta_h G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (5)$$

where $\beta^2{}_0 = N^2{}_0\nu^{-2}$, $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$, and $\delta(x)$ is the Dirac delta function. Understandably, here the primary field $\mathbf{Q}_0(\mathbf{r})$ and the Green's function $G(\mathbf{r}, \mathbf{r}')$ satisfy the necessary boundary conditions. Then it is possible to write an integral equation equivalent to the first equation of system (4) in the form

$$W(\mathbf{r}) = W_0(\mathbf{r}) + \int G(\mathbf{r}, \mathbf{r}')\beta_0^2(\mathbf{r}')\Delta_h[\mathbf{Q}(\mathbf{r}')\boldsymbol{\beta}(\mathbf{r}')]\,d\mathbf{r}'. \quad (6)$$

The series for function $W(\mathbf{r})$ is constructed by iterating the integral equation (6). The first term of the series is the primary field. The second term,

$$W_1(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\beta_0^2(\mathbf{r}')\Delta_h[\mathbf{Q}_0(\mathbf{r}')\boldsymbol{\beta}(\mathbf{r}')]\,d\mathbf{r}' \quad (7)$$

describes a singly scattered field. It is generated directly by interaction of the primary field $\mathbf{Q}_0(\mathbf{r})$ with the density inhomogeneity. We limit ourselves to a solution of (6) in the form $W_B(\mathbf{r}) = W_0(\mathbf{r}) + W_1(\mathbf{r})$ (Born approximation) by assuming the perturbation $\boldsymbol{\beta}(\mathbf{r})$ to be sufficiently small. For the solution found for (6), the equations for the functions U and V in the Born approximation are Poisson equations whose solutions have been studied well enough (see [10], for example). Hence, we limit ourselves below to a discussion of (7).

In order to clarify better the fundamental regularities of scattering, we make a number of assumptions to simplify the analysis, but meanwhile conserve a sufficient generality for many applications of the theory. The assumptions reduce to the following:

1. The medium is considered unbounded with $\beta^2{}_0$ = const.

2. The primary field is a plane monochromatic wave being propagated in the negative direction of the coordinate axes at an angle $0 < \alpha_0 < \pi/2$ to the horizontal plane

$$\mathbf{U}_0(r,\ t) = \mathbf{A}_0 \exp\ [i(\mathbf{kr} + \nu t)],$$

where **k** is the wave vector and the following dispersion relationship is satisfied:

$$\nu = N_0/\beta_0 = N_0 \cos \alpha_0. \tag{8}$$

3. The density field inhomogeneity is an isolated volume D with $\beta(\mathbf{r}) = \text{const}$.

To calculate $W_1(\mathbf{r})$, Green's function $G(\mathbf{r},\ \mathbf{r}')$ must be found. Within the framework of the assumptions made, the boundary condition of the problem here will be that $G(\mathbf{r},\ \mathbf{r}')$ tends to zero at infinity. We shall seek the solution of (5) in the form

$$G(x,\ y,\ z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} G(x,\ y,\ p)\, e^{-ipz} dp. \tag{9}$$

Substituting this expression into (5), we obtain an equation for the function G whose solution is known [11]:

$$G(x,\ y,\ p) = \frac{i}{4\beta_1^2} H_0^{(1)}(\gamma\rho),$$

$\beta^2_1 \equiv \beta^2_0 - 1$, $\gamma \equiv p/\beta$, $\rho = \sqrt{x^2 + y^2}$, $H^{(1)}_0(x)$ is the zeroth-order Hankel function of the first kind. Hence, the formula [12]:

$$G(\rho,\ z) = \begin{cases} \dfrac{1}{2\pi\beta_1 \sqrt{\beta_1^2 z^2 - \rho^2}}, & \beta_1^2 z^2 - \rho^2 > 0, \\ \dfrac{i}{2\pi\beta_1 \sqrt{\rho^2 - \beta_1^2 z^2}}, & \beta_1^2 z^2 - \rho^2 < 0 \end{cases} \tag{10}$$

can be obtained for Green's function $G(x,\ y,\ z)$. Substituting (10) for Green's function $G(\mathbf{r} - \mathbf{r}')$ into the formula for the scattered field W_1, we obtain

$$W_1(\rho,\ z) = -\frac{(\beta \mathbf{A}_0)\, k^2 \operatorname{ctg} \alpha_0}{2\pi} \left\{ \int\limits_D \frac{\exp(i\mathbf{kr})\, \theta\left[\beta_1^2 (z - z')^2 - (\rho - \rho')^2\right]}{\sqrt{\beta_1^2 (z - z')^2 - (\rho - \rho')^2}} d\rho' dz' + i \int\limits_D \frac{\exp(i\mathbf{kr})\, \theta\left[(\rho - \rho')^2 - \beta_1^2 (z - z')^2\right]}{\sqrt{(\rho - \rho')^2 - \beta_1^2 (z - z')^2}} d\rho' dz' \right\}, \tag{11}$$

where

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Let us transform (11). It is seen that the double integrals over $\boldsymbol{\rho}'$ are the convolutions $h_* q_1$ and $h_* q_2$, respectively, where

$$h(\rho,\ z) \equiv f(\rho,\ z) \exp\ [ik_\rho \rho],$$

$$q_1(\rho) \equiv \frac{\theta(p^2 - \rho^2)}{\sqrt{p^2 - \rho^2}}, \quad q_2(\rho) \equiv \frac{\theta(\rho^2 - p^2)}{\sqrt{\rho^2 - p^2}}, \quad f(\rho,\ z) = \begin{cases} 1, & \rho,\ z \in D, \\ 0, & \rho,\ z \notin D. \end{cases}$$

Turning to the Fourier transforms for the functions $h(\boldsymbol{\rho},\ z)$, $q_1(p,\ \boldsymbol{\rho})$, $q_2(p,\ \boldsymbol{\rho})$ and applying the convolution theorem [13], we obtain

$$W_1(\rho,\ z) = -\frac{i\ (\beta \mathbf{A}_0)\, k^2 \operatorname{ctg} \alpha_0}{4\pi^2} \int\limits_{-\infty}^{\infty} \tilde{h}(\varkappa_\rho,\ z') \frac{\exp[-i\,|p|\,|\varkappa_\rho| + ik_z z']}{|\varkappa_\rho|}\, e^{-ip\varkappa_\rho} d\varkappa_\rho dz',$$

where

$$\tilde{h}(\varkappa_\rho,\ z) = \int\limits_{-\infty}^{\infty} f(\rho,\ z) \exp\left(i\,(\mathbf{k}_\rho + \varkappa_\rho)\,\rho\right) d\rho.$$

The function $\tilde{h}(\varkappa_\rho, z)$ can be evaluated for a broad class of functions $f(\rho, z)$. In particular, many examples of such computations can be found in optics (the computation of Fraunhofer diffraction by holes of different kinds) [14].

It is seen from (12) that the scattered wave is not unimodal but contains a continuous spectrum of wave numbers which do not satisfy the dispersion relationship for the incident wave in the general case.

An an illustration of the utilization of (12), we consider density inhomogeneities of two kinds.

1. Rectangular Cylinder with Generator along the Axis. Let

$$f(\rho, z) = \theta(c^2 - z^2)\theta(R - |\rho|),$$

where c is half the cylinder altitude and R is the cylinder radius. Then an expression [14]

$$\tilde{h}(\varkappa_\rho, z) = 2\pi^2 R \frac{J_1(R|\mathbf{k}_\rho + \varkappa_\rho|)}{|\mathbf{k}_\rho + \varkappa_\rho|}\theta(c^2 - z^2)$$

can be obtained for $\tilde{h}(\varkappa_\rho, z)$, where $J_1(x)$ is a first-order Bessel function of the first kind. Substituting $\tilde{h}(\varkappa_\rho, z)$ into (12) and integrating with respect to z', we obtain

$$W_1(\rho, z) = -i(\beta A_0)k^2 R \operatorname{ctg}\alpha_0 \int_{-\infty}^{\infty} L(\varkappa_\rho, z)\frac{J_1(R|\mathbf{k}_\rho + \varkappa_\rho|)}{|\varkappa_\rho||\mathbf{k}_\rho + \varkappa_\rho|} e^{-i\rho\varkappa_\rho} d\varkappa_\rho; \tag{13}$$

$$L(\varkappa_\rho, z) = \theta(z - c)e^{-i\beta_1|\varkappa_\rho|z}\frac{c\sin\alpha_1}{\alpha_1} + \theta(-z - c)e^{-i\beta_1|\varkappa_\rho|z}\frac{c\sin\alpha_2}{\alpha_2}$$

$$+ \theta(c - |z|)\left\{e^{\frac{i}{2}\left(\alpha_1 + \alpha_2\frac{z}{c}\right)}\frac{c\sin\left[\alpha_1\frac{1 + \frac{z}{c}}{2}\right]}{\alpha_1} + e^{\frac{i}{2}\left(\alpha_2 + \alpha_1\frac{z}{c}\right)}\frac{c\sin\left[\alpha_2\frac{1 - \frac{z}{c}}{2}\right]}{\alpha_2}\right\}, \tag{14}$$

where $\alpha_1 = -c(k_z + \beta_1|\varkappa_\rho|)$, $\alpha_2 = c(k_z - \beta_1|\varkappa_\rho|)$. It follows from (13) that the amplitude of the field W_1 is maximal for forward scattering. The domain of values $\varkappa_\rho \approx -\mathbf{k}_\rho$ here yields the main contribution to the integral.

A necessary condition for applicability of the Born approximation is the condition of smallness of the scattered-wave amplitude $|W_1|/|W_0| \ll 1$. Let us use (13) to estimate the limits of applicability of the approximation under consideration. Considering the principal maximums of the functions $L(\varkappa_\rho, z)$ and $\tilde{h}(\varkappa_\rho, z)$ to yield the main contribution to the integral, we obtain

$$|W_1| \leqslant \pi D k^2 \operatorname{ctg}\alpha_0 |(\beta A_0)| \min\left\{\left|\frac{1 - k_z c}{\beta_1 c}\right|, \left|\frac{1{,}22}{R} - |\mathbf{k}_\rho|\right|\right\},$$

where $D = 2\pi R^2 c$ is the volume of the density inhomogeneity. The condition for applicability of the Born approximation hence takes the form

$$Dk^2 \operatorname{ctg}\alpha_0 \frac{|(\beta A_0)|}{A_{0z}} \min\left\{\left|\frac{\operatorname{ctg}\alpha_0(1 - k_z c)}{c}\right|, \left|\frac{1{,}22}{R} - k_\rho\right|\right\} \ll 1. \tag{15}$$

2. Rectangular Parallelepiped. Let

$$f(\rho, z) = \theta(a - |x|)\theta(b - |y|)\theta(c - |z|),$$

where $2a$, 2b, and 2c are the lengths of the parallelepiped edges. In this case, we have for the function $\tilde{h}(\varkappa_\rho, z)$ [14]

$$\tilde{h}(\varkappa_\rho, z) = 4\theta(c - |z|)\frac{\sin a(k_x + \varkappa_x)}{k_x + \varkappa_x}\frac{\sin b(k_y + \varkappa_y)}{k_y + \varkappa_y},$$

from which

$$W_1(\rho, z) = -\frac{2i}{\pi^2}k^2 \operatorname{ctg}\alpha_0(\beta A_0)\int_{-\infty}^{\infty}\frac{L(\varkappa_\rho, z)}{|\varkappa_\rho|}\frac{\sin a(k_x + \varkappa_x)}{k_x + \varkappa_x}\frac{\sin b(k_y + \varkappa_y)}{k_y + \varkappa_y}e^{-i\rho\varkappa_\rho}d\varkappa_\rho. \tag{16}$$

Even in this case the domain of values $\varkappa_\rho \approx -\mathbf{k}_\rho$ yields the main contribution to the integral while W_1 is maximal for forward-scattering. Let us estimate the limits of applicability of the Born approximation. Going over to polar coordinates in (16), we can obtain the condition desired:

$$Dk^2 \operatorname{ctg}\alpha_0 \frac{|(\beta A_0)|}{A_{0z}} \min\left\{\left|\frac{\operatorname{ctg}\alpha_0 (1-k_z c)}{c}\right|, \sqrt{\left|\frac{a^2+b^2}{a^2b^2} - k_\rho^2\right|}\right\} \ll 1, \tag{17}$$

where D = 8$a$bc is the volume of the inhomogeneity.

It is interesting to examine the particular case a, b → ∞ (a jump in density of thickness 2c). Letting a and b in (16) tend to infinity, and utilizing the formula

$$\lim_{\alpha\to\infty} \frac{\sin \alpha x}{x} = \pi\delta(x),$$

we obtain

$$W_1^{\mathrm{j}}(\rho, z) = -2ik\frac{(\beta A_0)}{\sin\alpha_0} L(-\mathbf{k}_\rho, z)\, e^{i\mathbf{k}_\rho \rho}. \tag{18}$$

As should have been expected, plane wave interaction with the density jump does not result in a change in modal composition. Only the transmitted and reflected waves exist, where the amplitudes of these waves are identical for $k_z c \ll 1$. Let us note that (18) can be obtained from (13) also since $W_1(\rho, z) \to W^j{}_1(\rho, z)$ as R → ∞. The condition for applicability of the Born approximation to describe plane wave interaction with a density jump has the following form, as is seen from (18):

$$\frac{kc}{\sin\alpha_0} \frac{|(\beta A_0)|}{A_{0z}} \ll 1. \tag{19}$$

The conditions (15), (17), and (19) presented for applicability of the Born approximation for density field perturbation domains of a particular kind can be obtained from the general estimates of the applicability of the single scattering approximation, which should be performed on the basis of (12) for $W_1(\rho, z)$. In conformity with the properties of the Fourier transformation, the function $\tilde{h}(\varkappa_\rho, z)$ evidently diminishes rapidly as $|\varkappa_\rho|$ grows for large R ($Rk_\rho \gg 1$), where R is the characteristic horizontal dimension. The integral (12) with respect to z' diminishes rapidly with the growth of $|\varkappa_\rho|$ for large R or c in exactly the same manner, where c is the characteristic vertical dimension of the inhomogeneity. We consequently have

$$\frac{|W_1|}{|W_0|} = \frac{k^2 \operatorname{ctg}\alpha_0}{4\pi^2} \frac{|(\beta A_0)|}{A_{0z}} \left| \int_{-\infty}^{\infty} \frac{e^{-i\rho\varkappa_\rho}}{|\varkappa_\rho|} \left\{ \int_{-\infty}^{\infty} \tilde{h}(\varkappa_\rho, z')\, e^{-i|\rho||\varkappa_\rho| + ik_z z'} dz' \right\} d\varkappa_\rho \right| \leqslant k^2 \operatorname{ctg}\alpha_0 \frac{|(\beta \cdot A_0)|}{A_{0z}} D\varkappa_0,$$

where $\varkappa_0 = \min\left\{\left|\frac{1}{R} - k_\rho\right|, \left|\frac{\operatorname{ctg}\alpha_0 (1-k_\rho c)}{c}\right|\right\}$ limits the spectral domain of variation of the horizontal wave numbers in which the amplitude of the scattered wave is substantially different from zero.

Therefore, in the general case we have the following condition for the applicability of the Born approximation:

$$k^2 \operatorname{ctg}\alpha_0 \frac{|(\beta A_0)|}{A_{0z}} D \min\left\{\left|\frac{1}{R} - k_\rho\right|, \left|\frac{\operatorname{ctg}\alpha_0 (1-k_z c)}{c}\right|\right\} \ll 1.$$

Let us now estimate the order of the angular dimension of the domain in which the amplitude of the scattered wave is essential. To do this we write the scattered wave in the spectral form

$$\widetilde{W}_1(\varkappa) = -ik^2 \operatorname{ctg}\alpha_0 (\beta A_0) \frac{F(\varkappa)}{|\varkappa_\rho|} \{\pi [\delta(\varkappa_z + \beta_1|\varkappa_\rho|) + \delta(\varkappa_z - \beta_1|\varkappa_\rho|)] + i\left[\frac{1}{\varkappa_z - \beta_1|\varkappa_\rho|} - \frac{1}{\varkappa_z + \beta_1|\varkappa_\rho|}\right]\}; \tag{20}$$

$$F(\varkappa) = \int_{-\infty}^{\infty} \tilde{h}(\varkappa_\rho, z)\, e^{i(k_z+\varkappa_z)z} dz, \quad \varkappa = \{\varkappa_\rho, \varkappa_z\}. \tag{20a}$$

It is seen from (20) that the spatial spectrum of the scattered wave has peaks at frequencies satisfying the dispersion relationship (8). As follows from (12), the function $\tilde{h}(\varkappa_\rho, z)$ has a smooth maximum for $\varkappa_\rho \approx -\mathbf{k}_\rho$. The function $F(\varkappa)$ is maximal here for $\varkappa_\rho \approx -\mathbf{k}_\rho$ and $\varkappa_z \approx -k_z$, i.e., for forward scattering. Let us introduce the polar θ and azimuthal φ angles corresponding to the direction $\varkappa$ relative to the direction $-\mathbf{k}$. Then the range of the angles θ and φ in which the scattered wave amplitude is substantially different from zero can be estimated from the following conditions:

For θ according to (12),

$$2k_\rho R \sin(1/2)\theta \leqslant 1 \quad \text{or} \quad \theta \leqslant 1/k_\rho R;$$

For φ, according to (20a), for $|\varkappa_\rho| \approx |\mathbf{k}_\rho|$ and $|k_z - \varkappa_z|c \approx 1$,

$$\cos\varphi \approx \frac{k^2 + k_\rho^2 + (k_z + \Delta\varkappa_z)^2 - \Delta\varkappa_z^2}{2k\sqrt{k_\rho^2 + (k_z + \Delta\varkappa_z)^2}} = \frac{ck + \sin\alpha_0}{\sqrt{(ck + \sin\alpha_0)^2 + \cos^2\alpha_0}}.$$

Therefore, the size of the domain in which the scattered wave amplitude is substantial enclosed the angle $\theta \approx 1/k_\rho R$ in the horizontal plane and the angle

$$\varphi \approx \arccos\left[\frac{ck + \sin\alpha_0}{\sqrt{(ck + \sin\alpha_0)^2 + \cos^2\alpha_0}}\right]$$

in the vertical plane. It follows from the estimates obtained that, in contrast to optics, internal wave scattering remains anisotropic in the vertical plane even in the case of fine-scale inhomogeneities ($k_z c \ll \sin\alpha_0$). The scattering angle φ here tends to $\varphi_{\max} = \pi/2 - \alpha_0$.

Therefore, internal wave scattering by density inhomogeneities results in an energy redistribution between different modes and could be one of the energy transmission mechanisms over the spectrum. It can be seen that the constraints made in the analysis of (7) on the kind of density field inhomogeneity, the boundary conditions, and the initial internal wave field can be weakened. In particular, the following problems are perfectly visible: internal wave scattering by continuously inhomogeneous density fluctuations, taking account of the influence of the ocean–atmosphere interface and the presence of the bottom (the change in boundary conditions), consideration of the scattering of a set of internal waves with multimodal structure, etc. In our opinion, the question of the necessity to take account of the effects of multiple scattering is more complex. In itself this problem can be solved, in principle, within the framework of system (1). However, the dynamical state of the medium that occurs because of multiple internal wave scattering cannot be described by the system (1), for instance in connection with the degeneration of part of the wave field into turbulence or for other reasons. Hence, questions of the relationships of discarded terms in the initial system (1) and corrections to a singly scattered field must be investigated carefully in taking account of multiple internal wave scattering.

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